The Reflection of Diverging Waves by a Gyrostatic Medium

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This paper furnishes the basis for a companion one, which discusses the possibility of describing material particles as localized oscillatory disturbances in a mechanical medium. If a medium is to support such disturbances it must reflect a part of the energy of a diverging spherical wave. It is here shown that this property is possessed by a medium, such as that proposed by Kelvin, in which the elastic forces are of gyrostatic origin. This is due to the fact that, for a small constant angular displacement of an element of this medium, the restoring torque, instead of being constant, decreases progressively with time.

Introduction

IN A companion paper[1] it is pointed out that it may be possible to describe the behavior of material particles as that of moving patterns of wave motion, provided a medium can be found which is capable of sustaining a localized oscillatory disturbance. In most media this is not possible, for the energy of the disturbance would be propagated away in all directions. Something special in the way of a medium is therefore called for. It must be capable of trapping the wave energy released from a central source. Kelvin proposed a mechanical medium, the equations of which, for small disturbances, were identical with those of Maxwell for free space. The medium derived its elasticity from gyrostats. He recognized that, for finite disturbances, the restoring torque depends on the time as well as the angular displacement. It is the present purpose to show that this time dependence imparts to his medium exactly the energy trapping property required.

The Gyrostatic Ether

The concept of an ether with stiffness to rotation originated with MacCullagh[2] in 1839, and was further developed by Kelvin[3] in 1888. MacCullagh showed that certain optical phenomena associated with reflection could not be represented by the elastic solid ether of Fresnel, but required for their mechanical representation a medium in which the potential energy is a function of what is now called the curl of the displacement. Fitzgerald[4] remarked in 1880 that its equations are identical with those of the electromagnetic

[1] R. V. L. Hartley, Matter, a Mode of Motion—this issue of the *Bell System Technical Journal*.

[2] Collected Works of James MacCullagh, Longmans Green & Co., London, 1880, p. 145.

[3] Mathematical and Physical Papers of Sir William Thomson, Vol. III, Art. XCIX, p. 436, and Art. C. p. 466.

[4] Phil. Trans. 1880, quoted by Larmor, Ether and Matter, Cambridge Univ. Press, 1900, p. 78.

theory of optics developed by Maxwell. This conclusion is confirmed in later discussion by Gibbs,[5] Larmor,[4] and Heaviside.[6]

Kelvin, apparently unaware of MacCullagh's work, was led by similar considerations to the same result. He went farther and devised a physical model which consisted of a lattice, the points of which were connected by extensible, massless, rigid rods in such a manner that the structure as a whole was incompressible and non-rigid. Each of these rods supported a pair of oppositely rotating gyrostats. By a gyrostat he meant a spinning rotor mounted in a gimbal so that it is effectively supported at its center of mass and can have its spin axis rotated by a rotation of the mounting. The resultant angular momentum of the rotors was the same in all directions.

This model, considered as a continuous medium, exhibits a stiffness to absolute rotation, the nature of which can be described by comparing it with the elasticity of a solid. A solid is characterized by a rigidity n such that small displacements u, v, w are accompanied by a stress tensor, one component of which is

$$n\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

For the ether model the corresponding component is

$$n\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 2n\varphi$$

where φ is a small angular displacement of the element about the z axis. More generally a small vector rotation $\overline{\Delta\varphi}$ is accompanied by a vector restoring torque per unit volume,

$$\overline{\Delta T} = -4n\overline{\Delta\varphi}. \tag{1}$$

The quantity $4n$ therefore represents a stiffness to angular displacement of the element.

In the appendix it is shown that the lattice of gyrostats, treated as a continuous medium, exhibits this kind of elasticity. It is also shown that for infinitesimal displacements, the medium is described by the wave equations (8a and 6a).

$$\nabla \times \left(\frac{\overline{T}}{2}\right) = \rho_0 \frac{\partial \bar{q}}{\partial t}, \tag{2}$$

$$\nabla \times \bar{q} = -\frac{1}{\eta_0}\frac{\partial}{\partial t}\left(\frac{\overline{T}}{2}\right), \tag{3}$$

[5] Collected Works of J. Willard Gibbs, Longmans Green & Co., New York 1928, Vol. II, p. 232.
[6] Heaviside, Electromagnetic Theory, Ernest Benn, Ltd., London, 1893, Vol. I, p. 226.

where ρ_0 is the constant density, η_0 is a generalized stiffness of the undisturbed medium, given by (7a), $\bar{q}$ is the vector velocity, and $\overline{T}$ is the torque per unit volume. In a plane wave $\bar{q}$ is normal to the direction of propagation. $\frac{\overline{T}}{2}$ is a tractive force per unit area in the direction of $\bar{q}$, which acts on a surface normal to the direction of propagation.

If, however, the amplitude is finite the equations become much more complicated. For present purposes we need consider only waves for which there is no component of velocity or torque in the direction of propagation, and we need consider only plane polarized waves for which the direction of the velocity is the same at all times and places. Also, as will appear below, we are concerned with the equations which describe a wave of infinitesimal amplitude which is superposed on a finite disturbance. This description need cover only infinitesimal ranges of time and position. It can therefore be expressed in terms of wave equations in which the constants of the medium have local instantaneous values which depend on the finite disturbance.

Subject to these restrictions it is shown in the appendix that (2) is to be replaced by (23a)

$$\nabla \times \left(\frac{\overline{T}}{2}\right) = \bar{l}_q \rho \frac{\partial q}{\partial t}, \tag{4}$$

where $\bar{l}_q$ is a unit vector in the fixed direction of the velocity, and ρ is an instantaneous local density, defined in terms of the finite disturbance by (20a). And, in place of (3), (22a)

$$\nabla \times \bar{q} = -\bar{l}_\varphi \frac{1}{\rho c^2}\left(\frac{\partial}{\partial t}\left(\frac{\overline{T}}{2}\right) + 2\frac{\partial f}{\partial t}\right), \tag{5}$$

where $\bar{l}_\varphi$ is a unit vector in the direction of the axis of rotation, ρ is again an instantaneous local density, c is an instantaneous local velocity derived in the usual way from ρ and an instantaneous local stiffness η, while f is a function defined by the relation, (13a),

$$\overline{T} = -\bar{l}_\varphi 4f(\varphi, t).$$

This function takes account of the fact that when the spin axis of the rotor is given a constant finite displacement, the restoring torque is not constant as in (1), but changes with time as the spin axis rotates toward the axis of displacement, and so reduces the component of the spin which is normal to the displacement axis and so is effective in producing stiffness. $-4\frac{\partial f}{\partial t}$ represents the rate of this change in torque for a fixed angular displacement. $-4\frac{\partial f}{\partial \varphi}$ is to be interpreted as the rate of change of torque with angular

displacement, when the time consumed is infinitesimal, that is when the angular velocity is infinite. It is therefore an instantaneous local angular stiffness from which the instantaneous local generalized stiffness η is derived as in (19a).

To simplify these expressions, let the direction of propagation be x and that of q be y. Then

$$\nabla \times \bar{q} = i \frac{\partial}{\partial x} (jq) = k \frac{\partial q}{\partial x},$$

so l_φ is in the direction of z, and represents a clockwise rotation about z. (5) then becomes the scaler equation

$$\frac{\partial q}{\partial x} = - \frac{1}{\rho c^2} \left[\frac{\partial}{\partial t} \left(\frac{T}{2} \right) + 2 \frac{\partial f}{\partial t} \right]. \tag{6}$$

T is also in the z direction, so

$$\nabla \times \left(\frac{\bar{T}}{2} \right) = i \frac{\partial}{\partial x} \left(k \frac{T}{2} \right) = - j \frac{\partial}{\partial x} \left(\frac{T}{2} \right).$$

But $\bar{q}$ is in the y direction, so

$$\frac{\partial}{\partial x} \left(\frac{T}{2} \right) = - \rho \frac{\partial q}{\partial t}. \tag{7}$$

These, then, are the desired equations of motion, for the type of wave under consideration.

The Generation of Reflected Waves

In this section we shall show that when a finite wave is propagated in this medium each element of the medium becomes the source of auxiliary waves which propagate in both directions from the source.

To do this we shall make use of the argument by which Riemann[7] showed that this does not occur for sound waves in an ideal gas. This will first be restated in more modern language. We consider a plane wave propagating along the x axis. We picture the finite pressure p and the longitudinal velocity u at a point in the medium as having been built up by the successive superposition of waves of infinitesimal amplitude, each propagating relative to the medium in its condition at the time of its superposition. If the first increment is propagating in the positive direction,

$$du = \frac{dp}{\rho c},$$

[7] Lamb, Hydrodynamics, Sixth Edition, p. 481. Rayleigh, Theory of Sound, Second Edition, Vol. II, p. 38.

where the characteristic resistance is ρc. Here

$$c^2 = \frac{dp}{d\rho}.$$

He assumes adiabatic expansion, so that p and c are functions of ρ only. If a second incremental wave of pressure dp, also traveling in the positive direction, be added, its velocity increment, being relative to the medium, will add to that already present. Its value will be related to dp through a new characteristic resistance corresponding to the modified density resulting from the previous increment. Hence the velocity u resulting from a large number of such waves will be

$$u = \int_0^p \frac{dp}{\rho c} = w,$$

where w is the quantity represented by ω in Lamb's version. If, then, all of the wave propagation is in the positive direction

$$u = w.$$

Similarly, if an incremental wave is traveling in the negative direction,

$$du = \frac{-dp}{\rho c},$$

and the condition for all the propagation to be in that direction is

$$u = -w.$$

Obviously, then, if u has some other value than one of these it results from the addition of increments some of which propagate in each direction.

Riemann deduces from the aerodynamic equations that

$$\left(\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right)(w + u) = 0, \tag{8}$$

$$\left(\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right)(w - u) = 0, \tag{9}$$

That is, the value of $w + u$ is propagated in the positive direction with a velocity of $c + u$ and that of $w - u$, in the negative direction with a velocity $c - u$. If, over a finite range of x, a disturbance be set up such that neither of these quantities is zero, it must be made up of incremental waves in both directions. However, as $w + u$ propagates positively it will be accompanied at any instant by a value of $w - u$ which has been propagated from the other direction. But, since the value of this was initially finite over a limited distance only, when all of this finite range is passed, $w - u$ will be zero, u will

be equal to w and all of the wave will be traveling positively. A similar argument applies at the negative side of the wave. Thus the initial disturbance breaks up into two parts which travel in opposite directions without reflection. More generally, these considerations hold for any medium in which the stress is a function of the strain only.

For the ether model, since we have assumed the displacements are normal to the direction of propagation, the velocity of wave propagation relative to the medium is the same as that relative to the axes.

If now, following Riemann, we let

$$dw = \frac{1}{\rho c}\, d\left(\frac{T}{2}\right), \tag{10}$$

so that now

$$w = \int \frac{1}{\rho c}\, d\left(\frac{T}{2}\right),$$

then from (7) and (6)

$$\frac{\partial q}{\partial t} = -c\,\frac{\partial w}{\partial x},$$

$$\frac{\partial w}{\partial t} = -c\,\frac{\partial q}{dx} - \frac{2}{\rho c}\,\frac{\partial f}{dt}.$$

Adding and subtracting gives

$$\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)(w + q) = -\frac{2}{\rho c}\,\frac{\partial f}{\partial t},$$

$$\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right)(w - q) = -\frac{2}{\rho c}\,\frac{\partial f}{\partial t},$$

which are to be compared with (8) and (9). Hence when $\frac{\partial f}{\partial t}$ is not zero the values of $w + q$ and $w - q$ are not propagated without change.

To show that reflection occurs, consider a disturbance at a point x at time t, characterized by q and w. At x and $t + \Delta t$, $w + q$ will differ from the value it had at $x - c\Delta t$, t, or $w + q - \frac{\partial}{\partial x}(w + q)c\Delta t$, by $-\frac{2}{\rho c}\frac{\partial f}{\partial t}\Delta t$. The increment at x in time Δt is

$$\Delta w + \Delta q = -\frac{\partial}{\partial x}(w + q)c\Delta t - \frac{2}{\rho c}\,\frac{\partial f}{\partial t}\,\Delta t,$$

and

$$\Delta w - \Delta q = \frac{\partial}{\partial x}(w - q)c\Delta t - \frac{2}{\rho c}\frac{\partial f}{\partial t}\Delta t.$$

From which

$$\Delta w = -c\frac{\partial q}{\partial x}\Delta t - \frac{2}{\rho c}\frac{\partial f}{\partial t}\Delta t,$$

$$\Delta q = -c\frac{\partial w}{\partial x}\Delta t.$$

Hence the velocity is the same as when $\frac{\partial f}{\partial t}$ is zero but w is changed by $-\frac{2}{\rho c}\frac{\partial f}{\partial t}\Delta t$. But the only way in which w can change with q constant is by adding waves of equal amplitude propagating in opposite directions, so that their contributions to w are equal and those to q are equal and opposite. From (10) this involves an increment of $\frac{T}{2}$ of $-2\frac{\partial f}{\partial t}\Delta t$ or a time rate of change of $-2\frac{\partial f}{\partial t}$. This agrees with (6), from which it is evident that the presence of $\frac{\partial f}{\partial t}$ alters $\frac{\partial q}{\partial x}$ from what it would otherwise be by $-\frac{2}{\rho c^2}\frac{\partial f}{\partial t}$. But, since q is unchanged, the velocities at $x + \frac{\Delta x}{2}$ and $x - \frac{\Delta x}{2}$ are increased by $-\frac{1}{\rho c^2}\frac{\partial f}{\partial t}\Delta x$ and $\frac{1}{\rho c^2}\frac{\partial f}{\partial t}\Delta x$. The first is the velocity associated with an auxiliary wave which propagates in the positive direction of x, and the second that of one which propagates in the negative direction, that is a reflected wave. Hence the medium generates a reflected wave of $\frac{1}{\rho c^2}\frac{\partial f}{\partial t}$ per unit length in the direction of propagation.

The Reflection of a Progressive Diverging Wave

So far attention has been confined to a single point. If a continuous disturbance is being propagated, it is important to know how the waves reflected at different points combine, for it is conceivable that they may interfere destructively. From the standpoint of the application to be made of these results in a companion paper, the case of most interest is that in which energy is propagated outward from a central generator as a sinusoidal wave of finite amplitude, beginning at time zero. Near the center, the wave of displacement will include radial as well as tangential components. As the radius

increases the radial components hecome relatively negligihle. We shall confine our attention to this outer region, where, in the ahsence of reflection, the propagation differs from that of a plane wave only in that the amplitude varies inversely as the radius. We shall neglect the effect of any reflections on the outgoing wave, and calculate the resultant reflected wave at a radius r_1 as a function of the time and so of the radial distance r the wave front has traveled.

If the outgoing wave were of infinitesimal amplitude, its velocity q_0 could he represented by

$$q_0 = \frac{r_0}{r} Q_0 \sin (\omega t - kr), \tag{11}$$

for values of $r < ct$, and by zero for $r > ct$, where Q_0 is the amplitude at some reference radius r_0. Tbe sine function is chosen to avoid the necessity of an infinite acceleration at the wave front, as would he required by a cosine function. When the amplitude is finite this wave suffers distortion due to the fact that k which is equal to $\frac{\omega}{c}$ varies slightly with the variations in the instantaneous value of c. However, these will be small and, since fluctuations in velocity alone do not cause reflection, we shall neglect them. The procedure is to make use of q_0 to calculate the reflected wave increment generated in a length $\Delta r'$ at a radius r', calculate the amplitude and phase of this at a fixed point $r_1 < r'$, and at r_1 integrate the waves received there for values of r' from r_1 to the farthest point from which reflected waves can reach r_1 at the time t under consideration.

To find the reflected wave generated in a lengtb $\Delta r'$ at r', we have from above that its velocity

$$\Delta q' = \frac{1}{\rho c^2} \frac{\partial f}{\partial t} \Delta r'.$$

From (21a), (19a) and (17a)

$$\frac{1}{\rho c^2} = \frac{F_1'}{\eta_0 \left(1 - a\left[\int \varphi \, dt\right]^2\right)},$$

where η_0 and a are constants of the medium given by (7a) and (15a). From (18a)

$$\frac{\partial f}{\partial t} = -a\eta_0 \varphi^2 \int \varphi \, dt,$$

$$\frac{dq'}{dr'} = -\frac{aF_1'\varphi^2 \int \varphi\, dt}{1 - a\left[\int \varphi\, dt\right]^2}$$

which reduces to

$$\frac{dq'}{dr'} = -a\varphi^2 \int \varphi\, dt,$$

if we neglect second powers of the variables compared with unity.

To the same accuracy, from (14a)

$$\varphi = \frac{1}{2}\int \frac{\partial q_0}{\partial r'}\, dt.$$

From (11)

$$\frac{\partial q_0}{\partial r'} = -\frac{r_0 Q_0}{r'}\left[k \cos(\omega t - kr') + \frac{1}{r'}\sin(\omega t - kr')\right].$$

Here k is 2π over the wavelength so, if as we have assumed r_1, and therefore also r', is large compared with the wavelength, we may neglect the second term. Then

$$\varphi = -\frac{r_0 Q_0}{2cr'}\sin(\omega t - kr'),$$

$$\int \varphi\, dt = \frac{r_0 Q_0}{2c\omega r'}\cos(\omega t - kr'),$$

$$\frac{dq'}{dr'} = -\frac{a}{8\omega}\left(\frac{r_0 Q_0}{cr'}\right)^3 \sin^2(\omega t - kr')\cos(\omega t - kr'),$$

$$= -\frac{a}{8\omega}\left(\frac{r_0 Q_0}{cr'}\right)^3 [\cos(\omega t - kr') + \cos 3(\omega t - kr')].$$

This, when multiplied by $\Delta r'$, gives the value at r' of the wave, generated in the interval $\Delta r'$, which propagates in the negative direction of r. This is made up of components of frequency ω and 3ω. We are primarily interested, from the stand-point of reflection, in that of frequency ω, so we shall confine our attention to this component, with the understanding that the other can be treated in exactly the same fashion. As the fundamental component propagates inward to r_1 it increases in amplitude in the ratio $\frac{r'}{r_1}$ and suffers a phase lag of $k(r' - r_1)$. If we call the resultant of all the reflected waves at r_1, q_1', then the contribution to q_1' of the wave generated at r' is

$$\Delta q_1' = -\frac{a}{8\omega}\left(\frac{r_0 Q_0}{c}\right)^3 \frac{1}{r_1 r'^2} \cos\,(\omega t + kr_1 - 2kr')\Delta r'.$$

This is to be integrated from r_1 to the farthest point from which a reflected wave has reached r_1 at the instant t under consideration. This point is at $\frac{1}{2}(r_1 + ct)$. So

$$q_1' = -\frac{a}{8r_1\omega}\left(\frac{r_0 Q_0}{c}\right)^3 \int_{r_1}^{\frac{1}{2}(r_1+ct)} \frac{1}{r'^2} \cos\,(\omega t + kr_1 - 2kr')\, dr'.$$

Here the integrand is a function of r' and t and the upper limit of integration is also a function of t. We therefore make use of the relation[8]

$$\frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \left(\frac{\partial}{\partial\alpha} f(x, \alpha)\right) dx + f(b, \alpha)\,\frac{db}{d\alpha} - f(a, \alpha)\,\frac{da}{d\alpha}.$$

Putting t for α, r' for x we have

$$\frac{dq_1'}{dt} = \frac{a}{8r_1}\left(\frac{r_0 Q_0}{c}\right)^3 \left[\int_{r_1}^{\frac{1}{2}(r_1+ct)} \frac{1}{r'^2} \sin\,(\omega t + kr_1 - 2kr') - \frac{2c}{\omega}\,\frac{1}{(r_1 + ct)^2}\right]$$

which, upon integration becomes,

$$\frac{dq_1'}{dt} = \frac{a}{8r_1}\left(\frac{r_0 Q_0}{c}\right)^3 \left(\frac{1}{r_1} si(\omega t + kr_1) - 2k[Si(\omega t - kr_1) - Si(2kr_1)]\right.$$
$$\cdot \sin\,(\omega t + kr_1) - [Ci(\omega t + kr_1) - Ci(2kr_1)]$$
$$\left.\cdot \cos\,(\omega t + kr_1) - \frac{2c}{\omega(r_1 + ct)^2}\right).$$

Since q_1' is zero when t is $\frac{r_1}{c}$, its value at t will be found by integrating from $\frac{r_1}{c}$ to t, so

$$q_1' = \frac{a}{8r_1^2\omega}\left(\frac{r_0 Q_0}{c}\right)^3 \left(-\cos\,(\omega t - kr_1) + \frac{2r_1}{r_1 + ct} - 2kr_1\right.$$
$$\cdot \left[\omega \int_{r_1/c}^{t} Si(\omega t + kr_1) \sin\,(\omega t + kr_1)\, dt + Si(2kr_1)\right.$$
$$\cdot [\cos\,(\omega t + kr_1) - \cos 2kr_1] - \omega \int_{r_1/c}^{t} Ci(\omega t + kr_1) \cos\,(\omega t + kr_1)\, dt$$
$$\left.\left. + Ci(2kr_1)\,[\sin(\omega t + kr_1) - \sin 2kr_1]\right]\right).$$

which reduces to

[8] Byerly, Integral Calculus, second edition p. 99.

$$q_1' = -\frac{a}{8r_1^2\omega}\left(\frac{r_0 Q_0}{c}\right)^3 \Big(\cos(\omega t - kr_1) - \frac{2r_1}{r_1 + ct} + 2kr_1$$
$$\cdot [-[Si(\omega t + kr_1) - Si(2kr_1)] \cos(\omega t + kr_1)$$
$$- [Ci(\omega t + kr_1) - Ci(2kr_1)] \sin(\omega t + kr_1)$$
$$+ Si(2\omega t + 2kr_1) - Si(4kr_1)]\Big).$$

The first term represents tbe value at r_1 of an outwardly moving wave in phase quadrature with the main wave. The second is a transient, the value of which is equal and opposite to that of the first term at the instant that the main wave passes r_1. The first two terms in tbe inner bracket are waves which propagate inward and so are to be regarded as reflections of the main wave. The last two terms represent a velocity which is zero when the main wave passes r_1, and subsequently oscillates about and approaches $\frac{\pi}{2} - \mathrm{Si}(4kr_1)$. Physically it appears to result from the particular form chosen for the main wave, which starts abruptly as a sine wave. The time integral of the impressed force, and so the applied momentum, has a component in one direction. Presumably if the main wave built up gradually these terms would be absent.

Returning to the reflected waves, their amplitudes are zero when the main wave passes r_1, after which they become finite. $Si(x)$ and $Ci(x)$ oscillate about and approach $\frac{\pi}{2}$ and zero respectively as x approaches infinity. Hence, as t increases indefinitely, the amplitudes of the reflected waves approach $\frac{\pi}{2} - Si(2kr_1)$ and $Ci(2kr_1)$. For the assumed large values of $2kr_1$ these quantities are small compared with unity. When multiplied by $2kr_1$ their variation is very slow. Hence the amplitudes vary roughly as $\frac{1}{r_1^2}$, and approach zero as the main wave at r_1 approaches an ideal plane one.

However, the significant fact is not that the reflected waves are small but that they are of finite magnitude. Because of this tbe main wave will not behave exactly as we assumed above, but will decrease slightly more rapidly with increasing radius. This should increase the reflection sligbtly, for the existence of the reflected wave is dependent on the decrease in amplitude with distance when the radius of curvature is finite.

To describe exactly what happens when the generator begins sending out waves from a central point would be hopelessly complicated, but we may form a general picture. In the early stages where tbe curvature is considerable, the reflected waves would be quite large and the main wave would be

correspondingly attenuated. The arrival of the reflected waves at the generator adds a reactive component to the impedance of the medium, as seen from the generator, which reduces the power delivered to the medium. Meanwhile energy is being stored as standing waves in the medium and the rate of flow of energy in the wavefront is decreasing. The energy in successive shells of equal radial thickness decreases with increasing r, instead of being uniform as it would be in the absence of reflection. In the limit it approaches zero, but as the rate of decrease depends on the curvature, the rate of approach also approaches zero. As the rate at which energy is stored and that at which it is carried outward at the wavefront both approach zero, the resistance which the medium offers to the generator approaches zero, and its impedance approaches a pure reactance.

The total energy stored in the medium depends on how the over-all attenuation of the main wave is related to its amplitude. If there were no attenuation, the impedance would remain a pure resistance, the energy in successive shells would all be the same, and the total energy would increase linearly with r, and so with the time, and approach infinity. If the attenuation were independent of r, the total energy would approach a finite value. The present case is intermediate between these, the attenuation being finite but approaching zero with increasing r. If we assume it to vary as some power of the amplitude of the velocity, then W. R. Bennett has shown that if this power is less than the first the total energy approaches a finite value. If it is equal to the first, the energy approaches infinity as $\log r$, and if it is greater than this, the power approaches infinity more rapidly. Until more is known as to the actual variation of amplitude with distance, nothing definite can be said about the limit of the total energy.

APPENDIX: EQUATIONS OF THE KELVIN ETHER

We are concerned with the wave properties of the model for wavelengths long enough compared with the lattice constant so that it may be regarded as a continuous medium. Its density is equal to the average mass of the gyrostats per unit volume. Its elastic properties are to be derived from the resultant of the responses of the individual gyrostats.

We shall therefore begin by considering the behavior of a single element, which is shown schematically in Fig. 1. Here the outer ring of the gimbal, which is rigidly connected with the lattice, lies in the $x\,y$ plane. The axis about which the inner ring rotates is in the x direction, and the spin axis C of the rotor is in the z direction. We wish to examine the effect of a small angular displacement φ of the lattice, that is, of the outer ring. If it is about x or z, it will, because of the frictionless bearings, make no change in the rotor. If it is about y it will produce an equal displacement of the spin axis

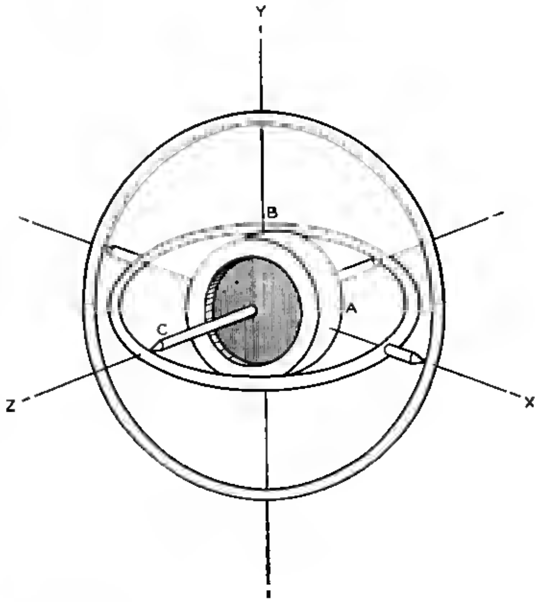

Fig. 1—Diagram of a gyrostat, showing its axes of rotation.

C about y. To study its effect we make use of Euler's equations for a rotating rigid body.[9]

$$A\,\frac{d\omega_1}{dt} - (B - C)\omega_2\omega_3 = L,$$

$$B\,\frac{d\omega_2}{dt} - (C - A)\omega_3\omega_1 = M,$$

$$C\,\frac{d\omega_3}{dt} - (A - B)\omega_1\omega_2 = N,$$

where ω_1, ω_2 and ω_3 are the angular velocities about three principal axes of inertia, fixed in the rotor, the moments of inertia about which are A, B and C, and L, M, and N are the accompanying torques about the three axes. They are also at any instant the values of the torques about that set of axes, fixed in space, which, at the instant, coincide with the axes 1, 2, 3, which are fixed relative to the body. We let the 3 axis coincide with the spin axis C. We choose as the 1 and 2 axes, lines in the rotor which, at the instant, are in the x and y directions respectively. Since the moments of

[9] Jeans, Theoretical Mechanics, Ginn and Co.. p. 308.

inertia about these are equal, A and B are equal. By virtue of the frictionless bearings the external torques L and N about 1 and 3 are zero.

Introducing these relations we have

$$A\frac{d\omega_1}{dt} + (C - A)\omega_2\omega_3 = 0, \tag{1a}$$

$$A\frac{d\omega_2}{dt} - (C - A)\omega_1\omega_3 = M, \tag{2a}$$

$$C\frac{d\omega_3}{dt} = 0. \tag{3a}$$

From (3a) the velocity of spin ω_3 remains constant. The torque M about y is then to be found from (1a) and (2a). For very small displacements,

$$\omega_2 = \dot{\varphi}.$$

Putting this in (1a) and integrating from zero to t, assuming φ to be zero at $t = 0$, gives

$$\omega_1 = -\frac{C - A}{A}\omega_3\varphi.$$

(2a) then becomes

$$A\ddot{\varphi} + \frac{(C - A)^2}{A}\omega_3^2\varphi = M.$$

This represents an angular inertia A and stiffness $\frac{(C - A)^2\omega_3^2}{A}$. The system will therefore resonate at a frequency $\frac{(C - A)\omega_3}{A}$. If the frequencies involved in the variation of φ are small compared with this, the inertia torque will be negligible, and the system will behave as a stiffness. If the displacements about A associated with ω_1 are very small the restoring torque M will act substantially about the y axis. That is, the lattice will encounter a stiffness to rotation.

Since the large number of gyrostats in an element of the model are oriented in all directions, an angular displacement of the lattice about y will generally not be about the B axis for each gyrostat. If it makes an angle α with this axis, then only the component $\varphi\cos\alpha$ of the angular displacement will be transmitted to the rotor. The resulting torque will then be $S\cos\alpha$, where

$$S = \frac{(C - A)^2\omega_3^2}{A}.$$

It will be directed about B and so will not be parallel to the applied displacement. However, if a second gyrostat has the position which the first

would have if it were rotated about y through π, its torque along y is the same as that of the first, and that normal to it is equal and opposite. Hence, if the gyrostats are properly oriented, the resultant torque will be parallel to the displacement and the medium will be isotropic. The y component of the opposing torque will be $S\varphi \cos^2 \alpha$. Thus if the B axes are uniformly distributed in space the total torque will be one third what it would be if they were all parallel to the axis of the applied displacement. Hence if there are N gyrostats per unit volume the vector restoring torque $\overline{T}$ per unit volume will be

$$\overline{T} = -\frac{N}{3} \frac{(C - A)^2 \omega_3^2}{A} \bar{\varphi}. \tag{4a}$$

The next step is to derive the wave equations for a medium having this stiffness to rotation. If the vector velocity $\bar{q}$ is very small,

$$\nabla \times \bar{q} = 2 \frac{\partial \bar{\varphi}}{\partial t}, \tag{5a}$$

where $\bar{\varphi}$ is a vector angular displacement of an element of the medium at the point under consideration. 2φ plays a role analagous with that of the dilatation in compressional waves. Then, from (4a) and (5a),

$$\nabla \times \bar{q} = -\frac{1}{\eta_0} \frac{\partial}{\partial t} \left(\frac{\overline{T}}{2}\right), \tag{6a}$$

where the generalized stiffness of the undisturbed medium,

$$\eta_0 = \frac{N}{12} \frac{(C - A)^2}{A} \omega_3^2. \tag{7a}$$

To get the companion equation, we interpret the torque exerted by an element in terms of the forces it exerts on the surfaces of neighboring elements. Let the x axis Fig. 2 be in the direction of the torque $T\Delta x^3$ which is exerted by the medium within the small cube. This very small torque can be resolved into the sum of two couples, one consisting of an upward force $F_y \Delta x^2$ on the right face and an equal downward force on the left one, and the other of a leftward force $F_z \Delta x^2$ on the upper surface and a rightward one on the lower one. But, if there is not to be a shearing stress, F_y and F_z must be equal, and each equal to $\frac{T}{2}$. Thus a torque per unit volume T is equivalent to a set of tangential surface forces per unit area of $\frac{T}{2}$ each.

Now consider the force exerted on an element by its neighbors, through the adjoining surfaces. To take the simplest case, let T in Fig. 2 be everywhere in the x direction and independent of z but varying with y. Then

the forces exerted on the upper and lower surfaces are equal and opposite. That downward on the right face exceeds that upward on the left by $\frac{\partial}{\partial y}\left(\frac{T}{2}\,\Delta x^2\right)\Delta y$, so the force in the z direction is $-\frac{\partial}{\partial y}\left(\frac{T}{2}\right)\Delta x^3$. By extending the argument to three dimensions it is easily shown that the total

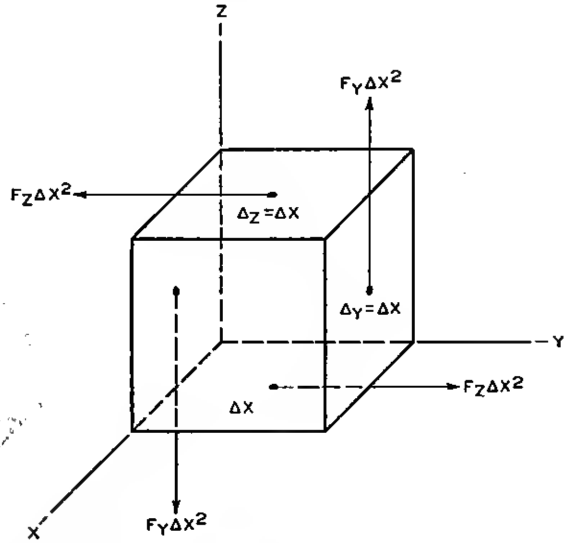

Fig. 2—Diagram showing the forces exerted by an element of the medium through its surfaces.

force is $\nabla \times \left(\frac{\overline{T}}{2}\right)\Delta x^3$. If ρ_0 is the density of the medium this force must equal $\rho_0 \Delta x^3 \frac{d\bar{q}}{dt}$, so

$$\nabla \times \left(\frac{\overline{T}}{2}\right) = \rho_0 \frac{d\bar{q}}{dt},$$

which, since $\bar{q}$ is small, reduces to

$$\nabla \times \left(\frac{\overline{T}}{2}\right) = \rho_0 \frac{\partial \bar{q}}{\partial t}. \tag{8a}$$

From this and (6a) the velocity of propagation is $(\eta_0/\rho_0)^{1/2}$ and the characteristic resistance is $(\rho_0\eta_0)^{1/2}$. In a plane wave the displacement is normal to the direction of propagation. The stress is a tractive force per unit area $\frac{T}{2}$ acting in a surface normal to the direction of propagation. It is in the direction of the velocity and in phase with it.

However, we are also interested in the case where the amplitudes are not negligible. We shall confine our attention to those cases where, as in plane or spherical waves at a distance from the source, the velocity is normal to the direction of propagation and the variations in the plane of the wave front are negligible. (5a) then becomes much more complicated. $\nabla \times \bar{q}$ is, however, still a function of $\frac{\partial \bar{\varphi}}{\partial t}$, say $2F_1\left(\frac{\partial \bar{\varphi}}{\partial t}\right)$. Then, for small variations of $\frac{\partial \bar{\varphi}}{\partial t}$ in the neighborhood of a particular value, we may write

$$\nabla \times \bar{q} = 2F_1'\left(\frac{\partial \bar{\varphi}}{\partial t}\right)\frac{\partial \bar{\varphi}}{\partial t} \tag{9a}$$

where $F_1'\left(\frac{\partial \bar{\varphi}}{\partial t}\right)$ is a function of the particular value of $\frac{\partial \bar{\varphi}}{\partial t}$. This relation is to take the place of (5a). Similarly, if

$$\nabla \times \left(\frac{\overline{T}}{2}\right) = F_2\left(\frac{\partial \bar{q}}{\partial t}\right),$$

then, in place of (8a), we are to use, for small variations,

$$\nabla \times \left(\frac{\overline{T}}{2}\right) = F_2'\left(\frac{\partial \bar{q}}{\partial t}\right)\frac{\partial \bar{q}}{\partial t}. \tag{10a}$$

When we come to the transition from (5a) to (6a), however, the situation is somewhat different. To see how this comes about, we go back to the behavior of the single gyrostat of Fig. 1. It was assumed above that the B axis coincided with the y axis. However, when the displacement of the rotor about A is finite, this is no longer exactly true. The situation is then as shown in Fig. 3. A rotation φ of the lattice about y displaces A in the xz plane by φ. The accompanying rotation of the rotor about A causes B to make an angle θ with y, which is independent of φ. Then

$$\omega_2 = \frac{d\varphi}{dt}\cos\theta.$$

From (1a)

$$\omega_1 = -\frac{C - A}{A}\,\omega_3 \int \frac{d\varphi}{dt}\cos\theta\, dt.$$

Also

$$\begin{aligned}\theta &= \int \omega_1\, dt,\\ &= -\frac{C - A}{A}\,\omega_3 \iint \frac{d\varphi}{dt}\cos\theta\, dt\, dt,\end{aligned} \tag{11a}$$

which determines θ as a function of φ and t. From (2a), neglecting the first term as above,

$$M = S \int \frac{d\varphi}{dt} \cos \theta \, dt,$$

and the restoring torque about y, or

$$T_y = -S \cos \theta \int \frac{d\varphi}{dt} \cos \theta \, dt. \tag{12a}$$

This, together with (11a), determines T_y as a function of φ and t, instead of φ alone as it is for infinitesimal displacements.

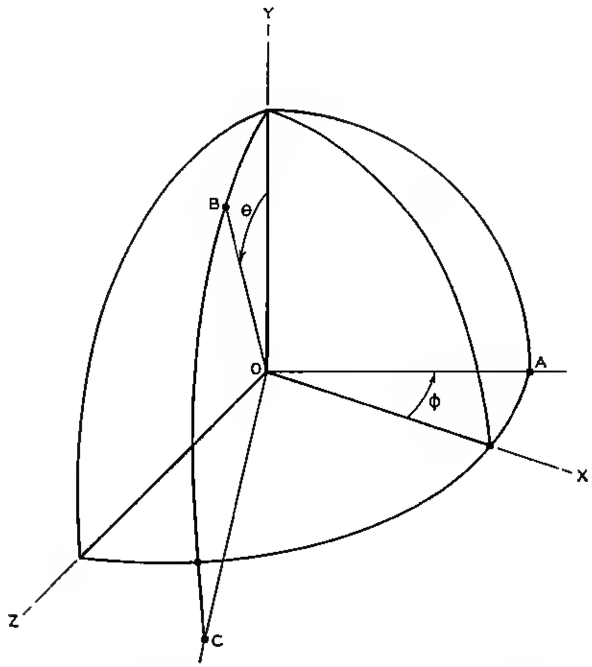

Fig. 3—Diagram showing the displacement of the axes of a gyrostat.

We assumed here that, in the rest position of the rotor, its B axis coincides with that of the applied displacement φ. When this is not the case, the relations are more complicated, but they should be qualitatively the same. Hence, for an element of the medium, the torque per unit volume should be a function of φ and t similar to T_y, which reduces to $-4\eta_0 \varphi$ for very small displacements. Since the restoring torque is in the direction of φ we may

write

$$\bar{T} = -\bar{l}_\varphi 4f(\varphi, t) \tag{13a}$$

where $\bar{l}_\varphi$ is a unit vector in the direction of the axis of rotation.

The derivation of the wave equation is much simpler if we consider only the case of present interest where the direction of the rotation is everywhere the same so that $\bar{l}_\varphi$ is constant. Then (9a) can be written as

$$\nabla \times \bar{q} = \bar{l}_\varphi 2F_1'\left(\frac{\partial \varphi}{\partial t}\right)\frac{\partial \varphi}{\partial t}, \tag{14a}$$

and (13a) as

$$T = -4f(\varphi, t).$$

We wish now to replace $\frac{\partial \varphi}{\partial t}$ by $\frac{\partial}{\partial t}\left(\frac{T}{2}\right)$. These partial derivatives refer to a constant position so we are interested in the total time derivatives of T as given by (12a). To get the desired relation we need to express T explicitly in terms of φ and t, that is, we must evaluate φ. Since the variables are small, we neglect their products of higher order than the third. Then

$$\cos\theta = 1 - \frac{1}{2}a\left[\int \varphi\, dt\right]^2,$$

where

$$a = \left(\frac{C - A}{A}\omega_3\right)^2. \tag{15a}$$

Putting

$$T = -4\eta_0 \cos\theta \int \frac{d\varphi}{dt}\cos\theta\, dt,$$

in accordance with (12a) and substituting for $\cos\theta$ gives

$$T = -4\eta_0\left[\varphi - a\varphi\left[\int \varphi\, dt\right]^2 + a\int \varphi^2\left(\int \varphi\, dt\right)dt\right].$$

Then

$$\frac{dT}{dt} = -4\eta_0\left[\left(1 - a\left[\int \varphi\, dt\right]^2\right)\frac{d\varphi}{dt} - a\varphi^2\int \varphi\, dt\right].$$

When φ is constant the first term is zero, so the second term can be interpreted as the partial derivative of T with respect to t. Physically this describes the change in torque for a fixed displacement which results from the

fact that, as the axis of the rotor rotates toward that of the applied torque, the component of the spin which is normal to the axis of displacement progressively diminishes. To interpret the first term, we let $\frac{d\varphi}{dt}$ increase indefinitely. The second term then becomes negligible, and when we divide through by $\frac{d\varphi}{dt}$, the left side becomes $\frac{dT}{d\phi}$. But the time increment which accompanies a finite increment of φ is now infinitesimal, and so this may be called the partial with respect to φ, with t constant.

We have then

$$\frac{dT}{dt} = -4\left(\frac{\partial f}{\partial \varphi}\frac{d\varphi}{dt} + \frac{\partial f}{\partial t}\right) \tag{16a}$$

where

$$\frac{\partial f}{\partial \varphi} = \eta_0\left(1 - a\left[\int \varphi\, dt\right]^2\right), \tag{17a}$$

$$\frac{\partial f}{\partial t} = -a\eta_0\varphi^2\int \varphi\, dt. \tag{18a}$$

Substituting for $\frac{\partial\varphi}{\partial t}$ from (16a) in (14a),

$$\nabla \times \bar{q} = -\bar{l}_\varphi \frac{F_1'}{\frac{\partial f}{\partial \varphi}}\left(\frac{\partial}{\partial t}\left(\frac{T}{2}\right) + 2\frac{\partial f}{\partial t}\right).$$

We may interpret $\frac{\partial f}{\partial \varphi}$ as an instantaneous stiffness to rotation and define an instantaneous local generalized stiffness by the relation

$$\eta = \frac{\frac{\partial f}{\partial \varphi}}{F_1'}. \tag{19a}$$

Similarly from (10a) we may define an instantaneous density by the relation

$$\rho = F_2'. \tag{20a}$$

Then we may speak of an instantaneous velocity c given by

$$c^2 = \frac{\eta}{\rho}, \tag{21a}$$

and an instantaneous characteristic resistance ρc. Then

$$\nabla \times \bar{q} = -\bar{l}_\varphi \frac{1}{\rho c^2} \left(\frac{\partial}{\partial t} \left(\frac{T}{2} \right) + 2 \frac{\partial f}{\partial t} \right). \tag{22a}$$

(10a) becomes

$$\nabla \times \left(\frac{\bar{T}}{2} \right) = \bar{l}_q \rho \frac{\partial q}{\partial t}, \tag{23a}$$

where $\bar{l}_q$ is a unit vector in the fixed direction of the velocity. These are the equations of motion which apply to a very small disturbance superposed on a finite disturbance.